MATH 2050 C Lecture 9 (Feb 9)

[No Lecture/tutorial on Feb 11 & 16.]

Last week...... ε-K def$^{\underline{n}}$ for limits & some examples

Questions

① $\lim(x_n)$ exists?

② How to compute $\lim(x_n)$?

§ Limit Theorems (Textbook § 3.2)

independent of n!

Def$^{\underline{n}}$: (x_n) is **bounded** if $\exists\, M > 0$ s.t.

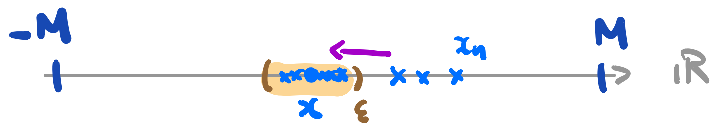
$$|x_n| \leq M \quad \forall\, n \in \mathbb{N}$$

Remark: This is equivalent to $\{x_n : n \in \mathbb{N}\}$ is bounded (as a set).

Thm: (x_n) convergent $\overset{\checkmark}{\Rightarrow}$ (x_n) bounded

Reason:

$(x_n) \to x$



Proof: Since (x_n) is convergent, by def$^{\underline{n}}$, $\exists\, x \in \mathbb{R}$ s.t.

$$\lim(x_n) = x.$$

[ie $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t. $|x_n - x| < \varepsilon \quad \forall n \geq K$]

Take $\varepsilon = 1$, by def$^{\underline{n}}$ of limit, $\exists\, K \in \mathbb{N}$ s.t.

$$|x_n - x| < \varepsilon = 1 \qquad \forall\, n \geq K$$

By Triangle ineq., $\forall n \geq K$.

$$|x_n| = |(x_n - x) + x| \leq |x_n - x| + |x| < 1 + |x|$$

Choose $M := \max\{|x_1|, |x_2|, \dots, |x_{K-1}|, 1 + |x|\} > 0$

Then. $|x_n| \leq M \quad \underline{\forall n} \in \mathbb{N}$

$\square$

Remark: The converse of this thm can be used to prove that a sequence is divergent.

i.e. (x_n) unbdd $\Rightarrow$ (x_n) divergent.

Example: $(x_n) := (n)$ unbdd, hence divergent.

Caution: (x_n) bdd $\not\Rightarrow$ (x_n) convergent

(we'll return to this later.)

Recall: $\mathbb{R}$ is a complete ordered field.

↖ ↑ ↗

(kind of) compatible

Limit Theorems: Suppose $\lim(x_n) = x$, $\lim(y_n) = y$. Then.

(i) $\lim(x_n \pm y_n) = x \pm y$

(ii) $\lim(x_n y_n) = xy$

(iii) $\lim\left(\frac{x_n}{y_n}\right) = \frac{x}{y}$, provided $y_n \neq 0 \ \forall n \in \mathbb{N}$ and $y \neq 0$

(i.e. the limits exist & are equal to the "expected" value.)

Proof: (i) Let $\varepsilon > 0$ be fixed but arbitrary.

Since $\lim(x_n) = x$ and $\lim(y_n) = y$,

$\exists\ K_1, K_2 \in \mathbb{N}$ s.t.

$$|x_n - x| < \frac{\varepsilon}{2} \qquad \forall\, n \geq K_1$$

and $$|y_n - y| < \frac{\varepsilon}{2} \qquad \forall\, n \geq K_2$$

Choose $K := \max\{K_1, K_2\} \in \mathbb{N}$, then $\forall\, n \geq K$, we have

$$|(x_n \pm y_n) - (x \pm y)| \overset{\because \Delta\text{-ineq.}}{\leq} |x_n - x| + |y_n - y|$$

$$\overset{\because K \geq K_1 \,\&\, K_2}{<} \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Goal: Find K s.t.

$$|\underbrace{(x_n+y_n)-(x+y)}_{\rotatebox{-90}{$\leq$}}| < \varepsilon$$

$$\underbrace{|(x_n - x)|}_{\substack{\text{small}\\ \because\, x_n \to x}} + \underbrace{|(y_n - y)|}_{\substack{\text{small}\\ \because\, y_n \to y}}$$

(ii) Let $\varepsilon > 0$ be fixed but arbitrary.

Since (y_n) is convergent, by previous thm,

(y_n) is bdd, ie $\exists\ M > 0$ s.t.

$$|y_n| \leq M \qquad \forall\, n \in \mathbb{N}$$

Take $M' := \max\{M, 1 + |x|\} > 0$

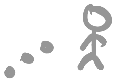
By def$^{\text{n}}$ of limit (taking $\varepsilon = \varepsilon/_{2M'} > 0$)

then $\exists\ K_1, K_2 \in \mathbb{N}$ s.t.

$$|x_n - x| < \frac{\varepsilon}{2M'} \qquad \forall\, n \geq K_1$$

and $$|y_n - y| < \frac{\varepsilon}{2M'} \qquad \forall\, n \geq K_2$$

Choose $K := \max\{K_1, K_2\} \in \mathbb{N}$, then $\forall\, n \geq K$, we have



Goal: Find K s.t. $\forall\, n \geq K$

$$|x_n y_n - xy| < \varepsilon$$

Estimate:

$$|x_n y_n - xy|$$

$$= |x_n y_n - x y_n + x y_n - xy|$$

$$\leq |y_n (x_n - x)| + |x (y_n - y)|$$

$$= \underbrace{|y_n|}_{\substack{\text{bdd}\\ \text{by Thm}}}\ \underbrace{|x_n - x|}_{\text{small}} + \underbrace{|x|}_{\substack{?\\ \text{fixed}\\ \text{number}}}\ \underbrace{|y_n - y}_{\text{small}}\ )$$

$$\leq M \underbrace{|x_n - x|}_{\varepsilon/2M} + M \underbrace{|y_n - y|}\ \varepsilon/2M$$

$$
\begin{aligned}
|x_n y_n - xy| &= |\underbrace{x_n y_n - xy_n} + \underbrace{xy_n - xy}| \\
&= |y_n(x_n - x) + x(y_n - y)| \\
&\le |y_n|\,|x_n - x| + |x|\,|y_n - y| \\
&\le M\,|x_n - x| + |x|\,|y_n - y| \\
&\le M'\,|x_n - x| + M'\,|y_n - y| \\
&< M' \cdot \frac{\varepsilon}{2M'} + M' \cdot \frac{\varepsilon}{2M'} = \varepsilon
\end{aligned}
$$

(iii) Since $\left(\frac{x_n}{y_n}\right) = \left(x_n \cdot \frac{1}{y_n}\right)$, using (ii), it suffices to show

$\underline{(\#)}$: $\lim\left(\frac{1}{y_n}\right) = \frac{1}{y}$ provided $y_n \neq 0$ $\forall n \in \mathbb{N}$ and $y \neq 0$.

Let $\varepsilon > 0$ be fixed but arbitrary.

We first establish a lemma.

<u>Lemma</u>: $\exists\, \tilde{K} \in \mathbb{N}$ st

$$|y_n| \ge \frac{|y|}{2} \qquad \forall\, n \ge \tilde{K}$$

<u>Pf of Lemma</u>: Since $\lim(y_n) = y$,

by take $\varepsilon := \frac{|y|}{2} > 0$ since $y \neq 0$.

$\exists\, \tilde{K} \in \mathbb{N}$ st $|y_n - y| < \varepsilon = \frac{|y|}{2}$

$\forall\, n \ge \tilde{K}$

By reverse triangle ineq, $\forall\, n \ge \tilde{K}$.

$$|y_n| = |y + (y_n - y)| \ge \big||y| - |y_n - y|\big|$$

$$\ge |y| - \frac{|y|}{2} = \frac{|y|}{2}.$$

<u>Goal</u>: Find K st $\forall n \ge K$

$$\left|\frac{1}{y_n} - \frac{1}{y}\right| < \varepsilon$$

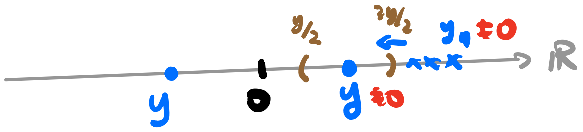
<u>Estimate</u>: (<u>WANT</u>: $|y_n - y|$)

$$\left|\frac{1}{y_n} - \frac{1}{y}\right| = \left|\frac{y - y_n}{y_n y}\right|$$

$$= \frac{|y_n - y|}{|y_n||y|}$$

$$= \boxed{\frac{1}{|y_n|}}\ \underbrace{\frac{1}{|y|}}_{\text{fixed no.}}\ \underbrace{|y_n - y|}_{\text{small}}$$

?????



<u>Lemma</u>: $|y_n| \ge \frac{|y|}{2} > 0 \quad \forall\, n \ge \tilde{K}$

Since $\lim(y_n) = y$. taking $\varepsilon / \frac{2}{|y|^2} > 0$.

$\exists\, K' \in \mathbb{N}$ s.t. $|y_n - y| < \varepsilon / \frac{2}{|y|^2} \quad \forall\, n \geq K'$ ⟵ (*)

Choose $K := \max\{\tilde{K}, K'\} \in \mathbb{N}$. then $\forall\, n \geq K$. we have

$$\left|\frac{1}{y_n} - \frac{1}{y}\right| = \left|\frac{y_n - y}{y_n y}\right| = \frac{1}{|y_n|}\cdot\frac{1}{|y|}\,|y_n - y|$$

$$\leq \underbrace{\boxed{\frac{1}{|y|/2}}}_{\text{by Lemma}} \cdot \frac{1}{|y|} \cdot \underbrace{\varepsilon / \frac{2}{|y|^2}}_{\substack{(*)\\ \text{choice of } K'}} = \varepsilon$$

$\square$

Non-example: The assumptions in (iii) are necessary.

Consider $(y_n) := (\frac{1}{n}) \to y = 0$. then

$$\left(\frac{1}{y_n}\right) = (n) \text{ is divergent.}$$

Remark: Converse of the Thm is not true.

Eg. $\underbrace{(x_n) = (\frac{1}{n})}_{\substack{\text{convergent}\\ \text{to } 0}}$, $\underbrace{(y_n) = (n)}_{\text{divergent}}$ then $\underbrace{(x_n y_n) = (1)}_{\text{convergent}}$

Thm: Let $(x_n), (y_n)$ be two sequences of real numbers s.t.

$$x_n \leq y_n \quad \forall\, n \in \mathbb{N} \qquad \text{———} \quad (**)$$

THEN. $\lim(x_n) \leq \lim(y_n)$ provided that their limits exist.

Remarks: (i) For (**), it is also sufficient to have

$$x_n \le y_n \quad \forall n \ge L \quad \text{for some fixed } L.$$

(ii) Even if we assume $x_n < y_n \ \forall n \in \mathbb{N}$ in (**),

we still get $\lim(x_n) \le \lim(y_n)$ only.

E.g.) $0 < \frac{1}{n} \quad \forall n \in \mathbb{N}$ But $\lim(0) = 0 = \lim(\frac{1}{n})$.

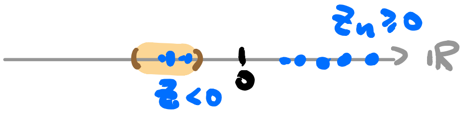
Proof of Thm: By Limit Thm (i), it suffices to show

(*) : $\underset{\text{conv. seq.}}{(z_n)}$ s.t. $z_n \ge 0 \ \forall n \in \mathbb{N} \Rightarrow \lim(z_n) =: z \ge 0$.

Suppose NOT., then $z := \lim(z_n) < 0$.

Take $\varepsilon = \frac{|z|}{2} > 0$, then $\exists\, K \in \mathbb{N}$ s.t.

$$|z_n - z| < \varepsilon = \frac{|z|}{2} \quad \forall n \ge K.$$



$$\Rightarrow \quad z_n < z + \frac{|z|}{2} = -\frac{|z|}{2} < 0 \quad \forall n \ge K.$$

Contradicting $z_n \ge 0 \ \forall n \in \mathbb{N}$!

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